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The Fermi pseudo-potential in one dimension

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Abstract
Wu and Yu recently examined point interactions in one dimension in the form of the Fermi pseudo-potential. On the other hand there are point interactions in the form of self-adjoint extensions (SAEs) of the kinetic energy operator. We examine the relationship between the point interactions in these two forms in the one-channel and two-channel cases. In the one-channel case the pseudo-potential leads to the standard three-parameter family of SAEs. In the two-channel case the pseudo-potential furnishes a ten-parameter family of SAEs.

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1. Introduction

Wu and Yu (WY) recently examined point interactions in one-dimensional quantum mechanics in the form of the Fermi pseudo-potential [1]. On the other hand there are point interactions that can be interpreted as self-adjoint extensions (SAEs) of the kinetic energy (KE) operator $-(\hbar^2/2m)\nabla^2$. For the SAEs, see, for example [2–5]. Many more papers have appeared on this subject. We will quote some of those papers in due course as they become relevant to the context of this paper. WY did not mention the SAE aspect of the point interactions. The purpose of this paper is to examine the relationship between the point interactions in the form of the pseudo-potential and the SAEs of the KE operator. We find that the Fermi pseudo-potential is a convenient device which enables us to obtain SAEs of the KE operator.

We consider one-channel and two-channel cases. In the one-channel case we find an explicit relation between the Fermi pseudo-potential and the standard three-parameter family of SAEs. In the two-channel case the pseudo-potential furnishes a ten-parameter family of SAEs of the KE operator. This result of the two-channel case goes beyond that obtained in a recent analysis [6].

We recapitulate WY's theory of the pseudo-potential in section 2 and some relevant aspects of the SAEs of the KE operator in section 3. In section 4 we examine the relationship between the pseudo-potential and the SAEs of the KE operator. In sections 2, 3 and 4 we assume that there is only one channel. In section 5 we discuss the case in which there are two coupled channels. The results are summarized in section 6. For notational brevity we take units such that $\hbar^2/(2m) = 1$ where m is the mass of the particle of the system under consideration.

2. Fermi pseudo-potential

Following WY we consider the time-independent Schrödinger equation in one dimension with a potential of a nonlocal form

$$-\psi''(x) + \int_{-\infty}^{\infty} \mathrm{d}x'\, V(x,x')\psi(x') = E\psi(x), \tag{1}$$

where $\psi''(x) = \mathrm{d}^2\psi(x)/\mathrm{d}x^2$. For $V(x,x')$ WY assumed the Fermi pseudo-potential

$$V(x,x') = g_1 v_1(x,x') + g_2 v_2(x,x') + g_3 v_3(x,x'), \tag{2}$$

where g_1, g_2 and g_3 are real constants and

$$\begin{aligned} &v_1(x,x') = \delta(x)\delta(x'), \qquad v_2(x,x') = \delta_p'(x)\delta(x') + \delta(x)\delta_p'(x'), \\ &v_3(x,x') = \delta_p'(x)\delta_p'(x'). \end{aligned} \tag{3}$$

The function $\delta_p'(x)$ is defined by

$$\delta_p'(x)\psi(x) = \delta'(x)\tilde{\psi}(x), \tag{4}$$

where $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$ and

$$\tilde{\psi}(x) = \begin{cases} \psi(x) - \frac{1}{2}(\psi_+ - \psi_-) & \text{for} \quad x > 0, \\ \psi(x) + \frac{1}{2}(\psi_+ - \psi_-) & \text{for} \quad x < 0. \end{cases} \tag{5}$$

Suffix + (−) refers to the boundary value for $x \to +0$ ($x \to -0$), e.g., $\psi_+ = \psi(+0)$. Note that function $\tilde{\psi}(x)$ is continuous at $x = 0$ and that

$$\tilde{\psi}(0) = \tfrac{1}{2}(\psi_+ + \psi_-). \tag{6}$$

It is understood that $\psi(x)$ can be discontinuous at $x = 0$, i.e., $\psi_+ \neq \psi_-$. In such a case, as we discuss in section 4, the product $\delta'(x)\psi(x)$ is ill-defined. The product $\delta'(x)\tilde{\psi}(x)$ with continuous $\tilde{\psi}(x)$, however, is well defined and so is $\delta_p'(x)\psi(x)$. Actually the $\tilde{\psi}(x)$ defined above is different from WY's $\tilde{\psi}(x)$ by an additive constant. We explain the reason for introducing this difference at the end of section 4.

WY solved the Schrödinger equation with the pseudo-potential and determined the resolvent operator for the equation. They further worked out the S-matrix for the transmission–reflection problem with the pseudo-potential. If a wave is incident from the left, the wavefunction can be written as [7]

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_L\,\mathrm{e}^{-\mathrm{i}kx} & \text{for} \quad x < 0 \\ T_L\,\mathrm{e}^{\mathrm{i}kx} & \text{for} \quad x > 0, \end{cases} \tag{7}$$

where $k > 0$ is related to the energy by $E = k^2$. The wavefunction of the case in which the wave is incident from the right can be written in a similar manner, with coefficients T_R and R_R. The S-matrix is a 2×2 matrix. It is related to T and R by

$$S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} = \begin{pmatrix} T_L & R_R \\ R_L & T_R \end{pmatrix}. \tag{8}$$

The $\pm$ of S_{++}, etc, unlike the $\pm$ of $\psi_\pm$ that we introduced in (5), refers to the direction of the wave propagation. When it is treated in the way as we explain in section 4, pseudo-potential (2) leads to

$$S = \left[\mathrm{i}g_3k + \tfrac{1}{2}\left(4 - g_1g_3 + g_2^2\right) + \mathrm{i}g_1k^{-1}\right]^{-1} \begin{pmatrix} \frac{1}{2}\left(4 + g_1g_3 - g_2^2\right) & \mathrm{i}g_3k + 2g_2 - \mathrm{i}g_1k^{-1} \\ \mathrm{i}g_3k - 2g_2 - \mathrm{i}g_1k^{-1} & \frac{1}{2}\left(4 + g_1g_3 - g_2^2\right) \end{pmatrix}. \tag{9}$$

This agrees with WY's S-matrix given by their (5.12) except that the signs of the terms with g_2 are all reversed. We, however, believe that (9) is correct. Time-reversal invariance holds so that $T_L = T_R$ [7]. If $g_2 = 0$ the left–right symmetry holds so that $R_L = R_R$. If $g_2 \neq 0$, then $R_L \neq R_R$. This is because $V(x, x') \neq V(-x, -x')$ in this case. In this connection, see the few lines below (25).

3. Self-adjoint extensions of the kinetic energy operator

An SAE of the KE operator can be represented by the following boundary condition which applies to any wavefunction $\psi(x)$ at $x = 0$ [8–10]:

$$\begin{pmatrix} \psi'_+ \\ \psi_+ \end{pmatrix} = U \begin{pmatrix} \psi'_- \\ \psi_- \end{pmatrix}, \qquad U = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \tag{10}$$

where $\psi' = \mathrm{d}\psi/\mathrm{d}x$. It is understood that $\psi(x)$ is twice differentiable except at $x = 0$. In general $\psi(x)$ and $\psi'(x)$ are discontinuous at $x = 0$. The phase θ and the matrix elements α, β, γ and δ are all real constants and are subject to the condition

$$\alpha\gamma - \beta\delta = 1. \tag{11}$$

Hence only three of α, β, γ and δ are independent. Condition (11) is necessary and sufficient for the self-adjointness of the KE operator. (We do not consider cases in which the two half-spaces of $x > 0$ and $x < 0$ are disjoint.)

The boundary condition (10) together with (11) represents a point interaction at the origin. There are four independent parameters including θ. As was pointed out in [11], however, θ is redundant. This is so as far as stationary problems are concerned. All that θ does is to introduce a constant phase difference between $\psi(x > 0)$ and $\psi(x < 0)$. Although the wavefunction depends on θ, observable quantities like the transmission and reflection probabilities, the energy eigenvalue and the probability density of a bound state are all independent of θ. Albeverio *et al* [12] pointed out that θ corresponds to a singular gauge field concentrated at the origin and can play a significant role in nonstationary problems. Such an aspect of nonstationary problems is beyond the scope of this paper. In many-body problems, θ may have subtle implications in relation to the symmetry of the wavefunction [13, 14], but we do not consider many-body problems in this paper either. If we require time-reversal invariance of the point interaction, U and hence $\mathrm{e}^{\mathrm{i}\theta}$ have to be real. In [8–10], θ was set to

$$\mathrm{e}^{\mathrm{i}\theta} = -1. \tag{12}$$

We follow this convention in the rest of this paper. In the abstract and section 1 we mentioned the 'standard' three-parameter family of SAEs, which is what we have just reviewed above. If the interaction is invariant under space reflection $x \to -x$, the boundary condition has to be invariant under $\psi_\pm \to \psi_\mp$ and $\psi'_\pm \to -\psi'_\mp$. This holds if and only if $\mathrm{e}^{\mathrm{i}\theta}$ is real and $\alpha = \gamma$.

Let us mention two special cases. For the usual δ-function potential $V(x) = g\delta(x)$ where g is a real constant, we obtain

$$U = -\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \tag{13}$$

i.e., $\alpha = -1$, $\beta = -g$, $\gamma = -1$ and $\delta = 0$. The $\psi'(x)$ is discontinuous at the origin but $\psi(x)$ is continuous.

The so-called δ'-interaction, which should not be confused with $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$, is defined in terms of

$$U = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}, \tag{14}$$

i.e., $\alpha = -1$, $\beta = 0$, $\gamma = -1$ and $\delta = -h$ [2, 8–10, 15]. Here h is a real constant. The $\psi(x)$ is discontinuous at the origin but $\psi'(x)$ is continuous. These two interactions are both invariant under space reflection.

The T and R of the transmission–reflection problem are given by (16)–(19) of [10],

$$T_L = T_R = -2\mathrm{i}k/D, \tag{15}$$

$$R_L = [\beta + k^2\delta + \mathrm{i}k(\alpha - \gamma)]/D, \tag{16}$$

$$R_R = [\beta + k^2\delta - \mathrm{i}k(\alpha - \gamma)]/D, \tag{17}$$

$$D = -\beta + k^2\delta + \mathrm{i}k(\alpha + \gamma). \tag{18}$$

See also (6)–(9) of [11].

4. Fermi pseudo-potential versus self-adjoint extensions of the kinetic energy operator

We can start with the Fermi pseudo-potential and derive the boundary condition on the wavefunction and its derivative at the origin. We utilize the following two prescriptions which we denote with $\mathcal{A}$ and $\mathcal{B}$, respectively:

$$\mathcal{A}[f(x)] \equiv \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, f(x), \tag{19}$$

$$\mathcal{B}[f(x)] \equiv \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}x \int_{-L}^{x} \mathrm{d}x'\, f(x'). \tag{20}$$

It is understood that $\epsilon > 0$ and $L > \epsilon$ and $f(x)$ is an arbitrary function of x. Applying the above prescriptions to the terms of the Schrödinger equation (1) we obtain

$$\mathcal{A}[\psi''(x)] = \psi'_+ - \psi'_- = \mathcal{A}\left[\int_{-\infty}^{\infty} \mathrm{d}x'\, V(x, x')\psi(x')\right], \tag{21}$$

$$\mathcal{B}[\psi''(x)] = \psi_+ - \psi_- = \mathcal{B}\left[\int_{-\infty}^{\infty} \mathrm{d}x'\, V(x, x')\psi(x')\right]. \tag{22}$$

If $V(x, x') = g_1\delta(x)\delta(x')$, which is equivalent to $V(x) = g_1\delta(x)$, the above manipulations are simple and we obtain the well-known boundary condition (13) with $g = g_1$.

At this point let us mention a complication that arises if one assumes the potential $V(x) = g\delta'(x) = g\,\mathrm{d}\delta(x)/\mathrm{d}x$ or $V(x, x') = g[\delta'(x)\delta(x') + \delta(x)\delta'(x')]$. Then appears the integral

$$\int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta'(x)\psi(x) = [\delta(x)\psi(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta(x)\psi'(x) \tag{23}$$

in prescription $\mathcal{A}$. If $\psi(x)$ is discontinuous at $x = 0$, $\psi'(x)$ contains $(\psi_+ - \psi_-)\delta(x)$. Thus the last integral of (23) becomes $(\psi_+ - \psi_-)\delta(0)$, i.e., it diverges. However, if we replace $\delta'(x)$ with $\delta'_p(x)$, we obtain

$$\int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta'_p(x)\psi(x) = \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta'(x)\tilde{\psi}(x)$$
$$= [\delta(x)\tilde{\psi}(x)]_{-\epsilon}^{\epsilon} - \int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta(x)\tilde{\psi}'(x) = -\frac{1}{2}(\psi'_+ + \psi'_-), \tag{24}$$

where $\psi'_\pm = \psi'(\pm 0)$. In prescription $\mathcal{B}$ we also need

$$\int_{-L}^{x} \mathrm{d}x'\, \delta'_p(x')\psi(x') = [\delta(x')\tilde{\psi}(x')]_{-L}^{x} - \int_{-L}^{x} \mathrm{d}x'\, \delta(x')\tilde{\psi}'(x')$$
$$= \frac{1}{2}[\delta(x)(\psi_+ + \psi_-) - \theta(x)(\psi'_+ + \psi'_-)], \tag{25}$$

where $\theta(x) = 1$ (0) for $x > 0$ ($x < 0$). We have used (6) and $\delta(-L) = 0$. It follows from (24) that $\delta'_p(x)$ is anti-symmetric, i.e., $\delta'_p(-x) = -\delta'_p(x)$. This leads to $v_1(x, x') = v_1(-x, -x')$, $v_2(x, x') = -v_2(-x, -x')$ and $v_3(x, x') = v_3(-x, -x')$.

In (24) and (25) we have used $\int_{-\epsilon}^{\epsilon} \mathrm{d}x\, \delta(x) f(x) = \frac{1}{2}(f_+ + f_-)$ which WY also used. This can be justified by replacing $\delta(x)$ with a smooth function of a finite width and then letting the width tend to zero. Griffiths and Walborn [16] remarked that this prescription involving $\delta(x) f(x)$ does not necessarily hold. But their remark does not apply to the present case. The $f(x)$ that they considered is not an arbitrarily given function. Rather it depends on ϵ in a certain specific manner. Such a situation arises, for example, in dealing with the δ-function potential in the one-dimensional Dirac equation [17, 18] but not in the situation that we are considering.

With $V(x, x')$ of (2) it is straightforward to work out (21) and (22), which respectively become

$$\psi'_+ - \psi'_- = \tfrac{1}{2}[g_1(\psi_+ + \psi_-) - g_2(\psi'_+ + \psi'_-)], \tag{26}$$
$$\psi_+ - \psi_- = \tfrac{1}{2}[g_2(\psi_+ + \psi_-) - g_3(\psi'_+ + \psi'_-)]. \tag{27}$$

The above can be written as

$$C_+ \begin{pmatrix} \psi'_+ \\ \psi_+ \end{pmatrix} = C_- \begin{pmatrix} \psi'_- \\ \psi_- \end{pmatrix}, \qquad C_\pm = \frac{1}{2}\begin{pmatrix} 2 \pm g_2 & \mp g_1 \\ \pm g_3 & 2 \mp g_2 \end{pmatrix}. \tag{28}$$

Then U is given by

$$U = C_+^{-1} C_- = \frac{1}{4\Delta}\begin{pmatrix} (2 - g_2)^2 - g_1 g_3 & 4g_1 \\ -4g_3 & (2 + g_2)^2 - g_1 g_3 \end{pmatrix}, \tag{29}$$

where

$$\Delta = \tfrac{1}{4}[(2 + g_2)(2 - g_2) + g_1 g_3]. \tag{30}$$

It is understood that $\Delta \neq 0$. Condition (11) is satisfied. Equation (29) together with (15)–(18) leads to the S-matrix of (9). Note that $\alpha = \gamma$ if and only if $g_2 = 0$. WY stated that 'That there are three parameters instead of two is a major surprise, ...': see a few lines below WY's equation (4.7). From the point of view of SAE, however, it is not surprising that point interactions can have three parameters.

If $g_1 \neq 0$ and $g_2 = g_3 = 0$, (29) is reduced to (13) with $g = g_1$, i.e., the U for the usual δ-function potential. If $g_3 \neq 0$ and $g_1 = g_2 = 0$, we obtain (14) with $h = g_3$, i.e., the U for

the δ' interaction. If $g_2 \neq 0$ and $g_1 = g_3 = 0$, we obtain $\alpha = 1/\gamma = (g_2 - 2)/(g_2 + 2)$ and $\beta = \delta = 0$. It is interesting (but not surprising) that these α, etc, of the $g_2 \neq 0$ case are the same as those Griffiths obtained by assuming $V(x) = c\delta'(x)$ where his c corresponds to our g_2 [16]. See (38) of [10] also. As pointed out in [10], however, there is a problem in Griffiths' derivation. It stems from what we pointed out below (23). Regarding the $\delta'(x)$ potential, [16] was quoted in [5] but the remark made in [16] is irrelevant with respect to the problem that we have just mentioned.

The definition of $\delta'_p(x)$ with which the pseudo-potential has been constructed may appear ad hoc and somewhat arbitrary. But it is justified by the fact that the pseudo-potential conforms to the SAE scheme of the KE operator. Below (5) we said that our $\tilde{\psi}(x)$ is different from the $\tilde{\psi}(x)$ that WY defined. Let us denote the latter with $\tilde{\psi}_{WY}(x)$. It is related to our $\tilde{\psi}(x)$ by

$$\tilde{\psi}_{WY}(x) = \tilde{\psi}(x) - \tilde{\psi}(0), \qquad \tilde{\psi}_{WY}(0) = 0. \tag{31}$$

This difference is unimportant in prescription $\mathcal{A}$ but is important in $\mathcal{B}$. (The method of calculation that WY used is different from ours. They did not use prescription $\mathcal{B}$.) If we assume $V(x) = c\delta'_p(x)$, which is equivalent to $V(x, x') = c[\delta'_p(x)\delta(x') + \delta(x)\delta'_p(x')] = cv_2(x, x')$, and use $\tilde{\psi}_{WY}(x)$ rather than $\tilde{\psi}(x)$ of (4), we are led to the boundary condition obtained by Zhao [20]. Zhao's boundary condition is not acceptable as pointed out in [10, 19, 21].

5. Two-channel case

We now consider the two-channel case. The situation that we have in mind is, there is a point object fixed at the origin, which is the source of a point interaction and which can be in two different states, 1 and 2. The particle that interacts with the point object can be described by means of a wavefunction that has two components [6]

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \tag{32}$$

Component 1 (2) is the wavefunction of the particle when the point object is in state 1 (2). WY advocated potential applications of such a two-channel system as a model of quantum memory but in this paper we focus on the mathematical aspect of the system. We assume the boundary condition at $x = 0$

$$\begin{pmatrix} \psi'_{1+} \\ \psi'_{2+} \\ \psi_{1+} \\ \psi_{2+} \end{pmatrix} = U \begin{pmatrix} \psi'_{1-} \\ \psi'_{2-} \\ \psi_{1-} \\ \psi_{2-} \end{pmatrix}, \qquad U = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \tag{33}$$

where $\psi'_{1+} = (\psi'_1)_+ = \psi'_1(+0)$, etc. Parameter θ is again redundant. We set it to $\mathrm{e}^{\mathrm{i}\theta} = -1$. The α, β, γ and δ are all 2×2 constant matrices and U a 4×4 matrix. In [6] it was assumed, for simplicity, that α, β, γ and δ are all Hermitian. Then it followed that the α, etc, all commute with each other. This time, however, let us not make that assumption and treat the α, etc, in their full generality. The reason for this generalization will become clear as we proceed: see the few lines below (49).

The α, etc, are subject to the following conditions:

$$\begin{aligned} &\alpha\beta^\dagger - \beta\alpha^\dagger = 0, \qquad &&\beta^\dagger\gamma - \gamma^\dagger\beta = 0, \\ &\delta\gamma^\dagger - \gamma\delta^\dagger = 0, \qquad &&\delta^\dagger\alpha - \alpha^\dagger\delta = 0, \end{aligned} \tag{34}$$

and

$$\gamma^\dagger\alpha - \beta^\dagger\delta = \sigma_0, \qquad \alpha\gamma^\dagger - \beta\delta^\dagger = \sigma_0, \tag{35}$$

where σ_0 is the 2×2 unit matrix. The above conditions are necessary and sufficient for the self-adjointness of the KE operator. The inverse U^{-1} is given by

$$U^{-1} = \mathrm{e}^{-\mathrm{i}\theta} \begin{pmatrix} \gamma^\dagger & -\beta^\dagger \\ -\delta^\dagger & \alpha^\dagger \end{pmatrix}. \tag{36}$$

If we assume invariance under space reflection, $x \to -x$, we obtain

$$\alpha = \gamma^\dagger, \qquad \beta = \beta^\dagger, \qquad \delta = \delta^\dagger. \tag{37}$$

Consider the Schrödinger equation of the form of

$$-\psi_1''(x) + \int_{-\infty}^{\infty} \mathrm{d}x' [V_{11}(x, x')\psi_1(x') + V_{12}(x, x')\psi_2(x')] = E\psi_1(x), \tag{38}$$

$$-\psi_2''(x) + \int_{-\infty}^{\infty} \mathrm{d}x' [V_{21}(x, x')\psi_1(x') + V_{22}(x, x')\psi_2(x')] = E\psi_2(x), \tag{39}$$

with the 2×2 potential matrix

$$V(x, x') = \begin{pmatrix} V_{11}(x, x') & V_{12}(x, x') \\ V_{21}(x, x') & V_{22}(x, x') \end{pmatrix}. \tag{40}$$

We require that $V(x, x')$ be Hermitian. Let us assume that the matrix elements of $V(x, x')$ are linear combinations of the $v_i(x, x')$ of (3). Then $V(x, x')$ in its most general form is given by

$$\begin{aligned} V_{11}(x, x') &= f_1 v_1(x, x') + f_2 v_2(x, x') + f_3 v_3(x, x'), \\ V_{12}(x, x') &= V_{21}^*(x, x') = g_1 \mathrm{e}^{\mathrm{i}\eta} v_1(x, x') + g_2 v_2(x, x') + g_3 v_3(x, x'), \\ V_{22}(x, x') &= h_1 v_1(x, x') + h_2 v_2(x, x') + h_3 v_3(x, x'), \end{aligned} \tag{41}$$

where f_i, g_i, h_i $(i = 1, 2)$ and η are all real constants. Thus we have ten parameters.

One may wonder whether or not one can obtain a more general situation by assuming $g_1 \mathrm{e}^{\mathrm{i}\eta_1}$, $g_2 \mathrm{e}^{\mathrm{i}\eta_2}$ and $g_3 \mathrm{e}^{\mathrm{i}\eta_3}$. This is not the case however. The part of the phase that is common among these three phases, which can be chosen as $\mathrm{e}^{\mathrm{i}\eta_3}$, can be eliminated by a constant unitary transformation of the Hamiltonian, leaving no physical effects. Recall that the g_2 part of the interaction is anti-symmetric, i.e., $v_2(x, x') = -v_2(-x, -x')$, while the other parts are symmetric. The g_2 part connects even-parity states to odd-parity states. The constant phase factor $\mathrm{e}^{\mathrm{i}\eta_2}$ can be absorbed into the relative phase between the even- and odd-parity states. This has no physical consequences.

Prescriptions $\mathcal{A}$ and $\mathcal{B}$ applied to the Schrödinger equation lead to

$$C_+ \begin{pmatrix} \psi_{1+}' \\ \psi_{2+}' \\ \psi_{1+} \\ \psi_{2+} \end{pmatrix} = C_- \begin{pmatrix} \psi_{1-}' \\ \psi_{2-}' \\ \psi_{1-} \\ \psi_{2-} \end{pmatrix}, \tag{42}$$

where

$$C_+ = \frac{1}{2} \begin{pmatrix} 2 + f_2 & g_2 & -f_1 & -g_1 \mathrm{e}^{\mathrm{i}\eta} \\ g_2 & 2 + h_2 & -g_1 \mathrm{e}^{-\mathrm{i}\eta} & -h_1 \\ f_3 & g_3 & 2 - f_2 & -g_2 \\ g_3 & h_3 & -g_2 & 2 - h_2 \end{pmatrix}, \tag{43}$$

and C_- is obtained from C_+ by reversing the signs of f_i, g_i and h_i simultaneously (but not the sign of η.) The U of the boundary condition at $x = 0$ is given by

$$U = -\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = C_+^{-1} C_-. \tag{44}$$

We observe that, when the signs of f_i, g_i and h_i are reversed in $U = C_+^{-1}C_-$, U is transformed to $U^{-1} = C_-^{-1}C_+$. Then α, β, γ and δ are respectively transformed to $\gamma^\dagger$, $-\beta^\dagger$, $\alpha^\dagger$ and $-\delta^\dagger$.

For illustration let us consider the case with $f_2 = g_2 = h_2 = 0$. The left–right symmetry holds in this case and the number of the parameters in the pseudo-potential is reduced from ten to seven. Let us write α as

$$\alpha = a_0\sigma_0 + \sum_{i=1}^{3} a_i\sigma_i, \tag{45}$$

and similarly for β, γ and δ with coefficients b_0 and b_i, etc. By working out (44) we obtain

$$\begin{aligned}
a_0 &= c_0 = -\big[16 - \big(f_1h_1 - g_1^2\big)\big(f_3h_3 - g_3^2\big)\big]/(16\Delta),\\
a_1 &= c_1^* = [g_3(f_1 + h_1) + g_1(f_3\,\mathrm{e}^{-\mathrm{i}\eta} + h_3\,\mathrm{e}^{\mathrm{i}\eta})]/(4\Delta),\\
a_2 &= c_2^* = \mathrm{i}[g_3(f_1 - h_1) - g_1(f_3\,\mathrm{e}^{-\mathrm{i}\eta} - h_3\,\mathrm{e}^{\mathrm{i}\eta})]/(4\Delta),\\
a_3 &= c_3^* = (f_1f_3 - h_1h_3 + 2\mathrm{i}g_1g_3\sin\eta)/(4\Delta),
\end{aligned} \tag{46}$$

$$\begin{aligned}
b_0 &= -\big[4(f_1 + h_1) + (f_3 + h_3)\big(f_1h_1 - g_1^2\big)\big]/(8\Delta),\\
b_1 &= -\big[4g_1\cos\eta - g_3\big(f_1h_1 - g_1^2\big)\big]/(4\Delta),\\
b_2 &= g_1\sin\eta/\Delta,\\
b_3 &= -\big[4(f_1 - h_1) - (f_3 - h_3)\big(f_1h_1 - g_1^2\big)\big]/(8\Delta),
\end{aligned} \tag{47}$$

$$\begin{aligned}
d_0 &= \big[4(f_3 + h_3) + (f_1 + h_1)\big(f_3h_3 - g_3^2\big)\big]/(8\Delta),\\
d_1 &= -\big[\big(f_3h_3 - g_3^2\big)g_1\cos\eta - 4g_3\big]/(4\Delta),\\
d_2 &= \big(f_3h_3 - g_3^2\big)g_1\sin\eta/(4\Delta),\\
d_3 &= \big[4(f_3 - h_3) - (f_1 - h_1)\big(f_3h_3 - g_3^2\big)\big]/(8\Delta),
\end{aligned} \tag{48}$$

where

$$16\Delta = 16 + 4(f_1f_3 + 2g_1g_3\cos\eta + h_1h_3) + \big(f_1h_1 - g_1^2\big)\big(f_3h_3 - g_3^2\big). \tag{49}$$

Note that a_i and c_i ($i = 1, 2, 3$) are generally complex while others are all real. Conditions (34) and (35) are satisfied. In [6] it was assumed for simplicity that α, etc, are all Hermitian. It is clear that this assumption does not hold in general.

Once α, etc, are obtained, the transmission and reflection coefficients, T and R, can be worked out. The T and R depend on all parameters including η. The seven parameters are all physically meaningful ones. We should add that the expressions for T and R given in [6] are in terms of Hermitian α, etc. When α, etc, are not necessarily Hermitian, those expressions have to be modified. In (36) for T_L and (39) for R_R of [6], all the α, etc, should be replaced by their respective Hermitian adjoints. Alternatively T_L and R_R can be obtained from T_R of (37) and R_L of (38) respectively, by substitutions: $\theta \to -\theta$, $\alpha \to \gamma^\dagger$, $\gamma \to \alpha^\dagger$, $\beta \to \beta^\dagger$ and $\delta \to \delta^\dagger$ (or equivalently by $\theta \to -\theta$, $\alpha \to \gamma^\dagger$, $\gamma \to \alpha^\dagger$, $\beta \to -\beta^\dagger$, $\delta \to -\delta^\dagger$ and $k \to -k$).

If we require that α, etc, be all Hermitian, we have to introduce constraints

$$\eta = 0, \qquad g_1(f_3 - h_3) - g_3(f_1 - h_1) = 0, \tag{50}$$

which leads to

$$a_2 = b_2 = c_2 = d_2 = 0, \qquad \frac{a_1}{a_3} = \frac{b_1}{b_3} = \frac{c_1}{c_3} = \frac{d_1}{d_3} = \frac{2g_1}{f_1 - h_1} = \frac{2g_3}{f_3 - h_3}. \tag{51}$$

Here it is understood that $f_1 - h_1 \neq 0$ and $f_3 - h_3 \neq 0$. If $f_1 - h_1 = 0$ and $f_3 - h_3 = 0$, then $a_3 = b_3 = c_3 = d_3 = 0$. The two constraints of (50) reduce the number of parameters from seven to five. This is completely consistent with what was found in [6].

We have assumed left–right symmetry in the above but foresee no difficulty in working out cases without such symmetry. The ten parameters that appear in such a general case are all physically meaningful ones.

Let us consider the time-reversal aspect of the system. It is understood that we deal with the time-dependent Schrödinger equation $\mathrm{i}\hbar\partial\psi(x,t)/\partial t = H\psi(x,t)$. The usual interpretation of time-reversal invariance is that, if $\psi(x,t)$ is a solution, so is $\psi^*(x,-t)$. If U is a real matrix, i.e., if $U = U^*$, then time-reversal invariance holds. In the general case that we examined above, U is not real. But U remains the same when its complex conjugate is taken and channels 1 and 2 are interchanged. If we define the time-reversal operation by

$$\psi(x,t) \to P_{12}\psi^*(x,-t), \tag{52}$$

where P_{12} interchanges channels 1 and 2, and if the Hamiltonian is free from other interactions that are affected by P_{12}, then the system is invariant under time reversal. This is the case if the pseudo-potential is the only interaction in the Hamiltonian. Let us add that, if there are three channels coupled, we can have more phase factors in the interaction. Then time reversal may be violated in nontrivial way. This aspect seems to have similarity to the time-reversal violation that arises in the Maskawa–Kobayashi model of the weak interactions [22].

As a possible model of quantum memory WY considered a special case of (41) in which all parameters other than f_3, g_1 and h_3 are set to zero. Furthermore they set $f_3 = -h_3$. For WY's model (46)–(49) are reduced to

$$\begin{aligned} &a_0 = c_0 = -[16 - (g_1 f_3)^2]/(16\Delta), \qquad a_1 = c_1^* = 0, \\ &a_2 = c_2^* = -\mathrm{i}g_1 f_3/(2\Delta), \qquad a_3 = c_3^* = 0, \end{aligned} \tag{53}$$

$$b_0 = 0, \qquad b_1 = -g_1/\Delta, \qquad b_2 = 0, \qquad b_3 = -g_1^2 f_3\big/(4\Delta), \tag{54}$$

$$d_0 = 0, \qquad d_1 = g_1 f_3^2\big/(4\Delta), \qquad d_2 = 0, \qquad d_3 = f_3/\Delta, \tag{55}$$

where

$$16\Delta = 16 + (g_1 f_3)^2. \tag{56}$$

Conditions (34) and (35) are satisfied. Unless $g_1 f_3 = 0$, the α and γ are non-Hermitian. The commutator $[\alpha,\gamma]$ vanishes but all other commutators between α, etc, are non-zero. This situation is different from that examined in [6]. (Note that WY's g_3 of this two-channel model corresponds to our f_3.)

Let us examine if unitarity is satisfied in the transmission–reflection problem of this model. If a wave is incident in channel 1 from the left, the wavefunction can be written as

$$\psi_1 = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_{L11}\,\mathrm{e}^{-\mathrm{i}kx} & \text{for} \quad x < 0 \\ T_{L11}\,\mathrm{e}^{\mathrm{i}kx} & \text{for} \quad x > 0, \end{cases} \tag{57}$$

$$\psi_2 = \begin{cases} R_{L21}\,\mathrm{e}^{-\mathrm{i}kx} & \text{for} \quad x < 0 \\ T_{L21}\,\mathrm{e}^{\mathrm{i}kx} & \text{for} \quad x > 0. \end{cases} \tag{58}$$

With the pseudo-potential with g_1 and f_3, we have left–right symmetry. Hence the T and R are independent of the direction of the incident wave and we can suppress subscript L. We obtain

$$T_{11} = \frac{8k^2 - \mathrm{i}g_1^2 f_3 k}{\left(g_1^2 + 4k^2\right)(2 + \mathrm{i}f_3 k)}, \tag{59}$$

$$R_{11} = \frac{-2g_1^2 + 4\mathrm{i} f_3 k^3}{\left(g_1^2 + 4k^2\right)(2 + \mathrm{i} f_3 k)}, \tag{60}$$

$$T_{21} = R_{21} = \frac{-2\mathrm{i} g_1 k}{g_1^2 + 4k^2}. \tag{61}$$

These coefficients satisfy

$$|T_{11}|^2 + |R_{11}|^2 + |T_{21}|^2 + |R_{21}|^2 = 1. \tag{62}$$

It is interesting that there is a bound state with the binding energy $\kappa_1^2 = g_1^2/4$, irrespective of the sign of g_1. If $f_3 > 0$ there is another bound state with the binding energy $\kappa_3^2 = 4/f_3^2$.

6. Summary and discussion

We have examined the relationship between the Fermi pseudo-potential à la WY and point interactions as SAEs of the KE operator in one dimension. The pseudo-potential is a convenient device which enables us to obtain the SAEs. We considered one-channel and two-channel cases. In the one-channel case, the Fermi pseudo-potential that WY developed contains three real parameters. The SAEs also contain, apart from the trivial parameter θ, three real parameters. We have found explicit relations between the three parameters of the pseudo-potential and those of the SAEs.

In passing let us mention that WY raised a question as to under what conditions Bethe's ansatz for a many-body system [23, 24] still holds when the δ-function is replaced by the Fermi pseudo-potential. See the first paragraph of part B of their paper [1]. This question was examined within the context of SAEs of the KE operator [13, 14, 25]. Let us also mention that the point interactions in the form of SAEs of the one-channel case can be interpreted in terms of renormalized short-range potentials as was pointed out by Cheon and Shigehara [26]. For more recent development in this connection see [27]. It would be interesting to examine the pseudo-potential, in particular, the definition of $\delta_p'(x)$, in the light of Cheon *et al*'s work and also extend such an analysis to the two-channel case.

In the two-channel case the pseudo-potential provides us with more general SAEs than obtained earlier [6]. This is in contrast to what was suggested towards the end of [6]. The two-channel pseudo-potential can have ten parameters, which means that the pseudo-potential can represent a ten-parameter family of SAEs. If the pseudo-potential is the only interaction that is contained in the Hamiltonian of the system, time-reversal invariance holds. Recall that, in [6], a family of SAEs with eight parameters (including θ_3) was obtained. If we impose space symmetry the number of the parameters is reduced from ten to seven. The corresponding number of the parameters that was obtained in [6] is five. When time-reversal invariance holds, the S-matrix of a two-channel system in one dimension can be expressed in terms of a 4×4 K-matrix that is real and symmetric. The S-matrix has ten real parameters. This is so irrespective of the form of the interaction as long as it conforms to time-reversal invariance. It is interesting that the number of the parameters of the S-matrix and that of the SAEs coincide.

As WY advocated the two-channel pseudo-potential can be a very powerful tool with which one can construct models of quantum memory, an essential component of quantum computing.

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